

Inner topological structure of Hopf invariant

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Abstract

In light of ϕ -mapping topological current theory, the inner topological structure of Hopf invariant is investigated. It is revealed that Hopf invariant is just the winding number of Gauss mapping. According to the inner structure of topological current, a precise expression for Hopf invariant is also presented. It is the total sum of all the self-linking and all the linking numbers of the knot family.

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I. INTRODUCTION

Results from the pure mathematical literature are of little difficulty for physicists and can be made accessible to physicists by introducing them in common physical methods. In this paper, we discuss an object from algebraic topology, Hopf invariant, and reveal the inner topological structure of Hopf invariant in terms of the so-called ϕ -mapping topological current theory.

Hopf[1] studied the third homotopy group of the 2-sphere and showed that this group $\pi_3(S^2)$ is non-trivial; later Hopf invented more non-trivial fibration and finally obtained a series of invariants $\pi_{2n-1}(S^n)$ (where n is positive integer) that now bear his name. In this paper, we will stick strictly to the Hopf map from 3-sphere(denoted as S^3) to 2-sphere(denoted as S^2) and the invariant $\pi_3(S^2)$ related to this map which can be expressed in the integral form[2]

$$H = \int_{S^3} \omega \wedge d\omega \ ,$$

where ω is a 1-form on S^3 . Hopf invariant is independent of the choice of 1-form ω , which leads to the normalization of ω .

Hopf invariant is an important topological invariant in mathematics and have many applications not only in condensed matter physics but also in high energy physics and field theory. A review article about Hopf fibration is appeared in [3] where the author pointed out Hopf fibration occurs in at least seven different situation in theoretical physics in various guises. Hopf invariant $\pi_3(S^2)$ have deep relationships with the Abelian Chern-Simons action[4, 5] in gauge field theory, self-helicity in magnetohydrodynamics(MHD)[6] and Faddeev-Niemi knot quantum number in Faddeev's model[7]. As revealed in this paper, Hopf invariant is an important topological invariant to describe the topological characteristics of the knot family. Knotlike configurations as string structures of finite energy(finite action) appear in a variety of physical, chemical, and biological scenarios, including the structure of elementary particles[7, 8], early universe cosmology[9], Bose-Einstein condensation [10], polymer folding[11], and DNA replication, transcription, and recombination[12], and have been taken

more and more attentions to.

The ϕ -mapping topological current theory[13] proposed by Prof.Duan is a powerful tool not only in studying the topological objects in physics, including vortex lines and monopoles in BEC[14], superfluid[15] and superconductivity[16], but also in discussing the topological objects in mathematics, for example, the inner topological structure of Gauss-Bonnet-Chern theorem[17], the second Chern characteristic class[18] and homotopy group. In this paper, in light of ϕ -mapping topological current theory, the inner topological structure of Hopf invariant is discussed in detail.

The main research interests of this paper are in the area of the inner topological structure of Hopf invariant and the precise expression for Hopf invariant. This paper is arranged as follows. In Sec.II, some basic mathematical ideas, including Hopf map, Hopf fibration and Hopf invariant, are briefly presented. By using the spinor representation of Hopf map, the inner topological structure of Hopf invariant $\pi_3(S^2)$ is discussed in detail which is revealed that Hopf invariant is the winding number of Gauss mapping $S^3 \rightarrow S^3$. In Sec.III, in light of ϕ -mapping topological current theory, an conserved topological current is introduced and its inner structure is also presented. In Sec.IV, a precise expression for Hopf invariant is obtained which is revealed that it is the total sum of all the self-linking and all the linking numbers of the knot family. A brief conclusion and prospect are appeared in the last section.

II. INNER TOPOLOGICAL STRUCTURE OF HOPF INVARIANT

In this section, we firstly introduce some basic mathematical ideas, including Hopf map, Hopf fibration and Hopf invariant. Then, in light of ϕ -mapping topological current theory, the inner topological structure of Hopf invariant $\pi_3(S^2)$ is revealed that Hopf invariant is just the winding number of Gauss mapping $S^3 \rightarrow S^3$.

The Hopf map $f : S^3 \rightarrow S^2$ arises in many contexts, and can be generalized to the map $S^{2n-1} \rightarrow S^n$, where n is an integer. There are several descriptions of the Hopf map. Here, the spinor representation is expected to discuss the inner topological structure of Hopf

invariant. So, firstly, we want to construct the spinor representation of Hopf map. Consider two complex scalar field z^1 and z^2 satisfying the condition

$$|z^1|^2 + |z^2|^2 = 1,$$

from which the two-component normalized spinor z can be introduced like this

$$z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}.$$

The normalized condition denotes that $z \in S^3$. In fact, it is easy to exhibit the basic Hopf map $S^3 \rightarrow S^2$. Generally, Hopf map can be represented by the spinor representation[19, 20]

$$m^a(x) = z^\dagger(x) \sigma^a z(x), \quad a = 1, 2, 3, \quad x \in R^3 \quad (1)$$

in which σ^a is Pauli matrix. Now, we will interpret why Eq.(1) is the Hopf map $S^3 \rightarrow S^2$. Physicists should be familiar with this fact from the elementary discussions of the Pauli matrices σ^a . Noticing that Pauli matrix elements satisfy the formula

$$\sigma_{\alpha\beta}^a \sigma_{\alpha'\beta'}^a = 2\delta_{\alpha\beta'} \delta_{\alpha'\beta} - \delta_{\alpha\beta} \delta_{\alpha'\beta'} \quad ,$$

one have $m^a m^a = 1$ which denotes that $m^a \in S^2$. The definition (1) does really not provide the Hopf map $S^3 \rightarrow S^2$ because of $x \in R^3$. Now, we add the boundary condition

$$\vec{m}(x)|_{|x| \rightarrow \infty} \mapsto \vec{m}_0. \quad (2)$$

where \vec{m}_0 is a fixed vector. It is to say that we assume the vector $\vec{m}(x)$ points to the same direction at spatial infinity, and therefore the spatial infinity can be efficiently contracted to a point, i. e. , $R^3 \rightarrow S^3$. Thus, now, the unit vector \vec{m} provides us the Hopf map $S^3 \rightarrow S^2$. Under the boundary condition (2), We will not distinguish between R^3 with S^3 .

In fact, we can take the Hopf map \vec{m} as a projection in the sense that S^3 is a principle fibre bundle(Hopf bundle or Hopf fibration) over the base space S^2 with the structure group $U(1)$. The standard fibre of Hopf bundle is S^1 which is the inverse image of the point of

S^2 under the Hopf map. In 3-sphere, S^1 is homeomorphous with knot and the quantity to describe the topology of these knots is Hopf invariant which is defined as[20]

$$H = \frac{1}{16\pi^2} \int A \wedge B, \quad (3)$$

where A is connection 1-form and $B = dA = \frac{1}{2}B_{ij}dx^i \wedge dx^j$ is the Hopf curvature 2-form[21] and the coefficient $1/16\pi^2$ is the normalized coefficient to ensure that H is an integer. According to Hopf map (1), the Hopf curvature can be constructed as

$$B_{ij} = \vec{m} \cdot (\partial_i \vec{m} \times \partial_j \vec{m}) = \epsilon_{abc} m^a \partial_i m^b \partial_j m^c. \quad (4)$$

Noticing that Pauli matrix elements satisfy the formula

$$\begin{aligned} & \epsilon_{abc} \sigma_{\alpha\beta}^a \sigma_{\alpha'\beta'}^b \sigma_{\alpha''\beta''}^c \\ &= -2i(\delta_{\alpha\beta'} \delta_{\alpha'\beta''} \delta_{\alpha''\beta} - \delta_{\alpha\beta''} \delta_{\alpha''\beta'} \delta_{\alpha'\beta}), \end{aligned}$$

one can arrive at the following expression for Hopf curvature after some algebra

$$B_{ij} = -2i(\partial_i z^\dagger \partial_j z - \partial_j z^\dagger \partial_i z).$$

Then obviously one can get

$$A_i = -2iz^\dagger \partial_i z. \quad (5)$$

which is just the canonical connection[3] of Hopf bundle. In the ϕ -mapping theory[13], since the spinor field z is the fundamental field which is essential to the topology properties of manifold itself, the canonical connection (5) just reveals the inner structure of Hopf bundle's connection. It is easy to see that A_i is in the form of $U(1)$ gauge potential and Hopf invariant is unchanged under the gauge transformation

$$A'_i = A_i + \partial_i \psi,$$

where ψ is an arbitrary complex scalar function. The invariance presents the $U(1)$ symmetry of Hopf fibration and is very important to the following discussion. Depending on the $U(1)$ invariance of Hopf invariant, we can select Columb gauge condition as done in classical electrodynamics, i.e. impose the condition $\partial_i A_i = 0$ in Sec.IV without losing generality.

In terms of the canonical connection form (5), Hopf invariant can be written as

$$H = \frac{1}{32\pi^2} \int \epsilon^{ijk} A_i B_{jk} d^3x = -\frac{1}{4\pi^2} \int \epsilon^{ijk} z^\dagger \partial_i z \partial_j z^\dagger \partial_k z d^3x.$$

Since the spinor field z is the fundamental field on manifold and just describes the topological property of the manifold itself, the above expression is obviously more direct in the study of Hopf invariant. The normalized two-component spinor z can be expressed by

$$z = \begin{pmatrix} l^0 + il^1 \\ l^2 + il^3 \end{pmatrix}, \quad (6)$$

where $l^a (a = 0, 1, 2, 3)$ is a real unit vector. In the ϕ -mapping theory[13] the unit vector l^a should be further determined by the smooth vectors φ^a , i.e.

$$l^a = \frac{\varphi^a}{\|\varphi\|}, \quad \|\varphi\| = \varphi^a \varphi^a. \quad (7)$$

Substituting the expression (6) of the two-component spinor z into Hopf invariant, one can get

$$H = \frac{1}{12\pi^2} \int \epsilon_{abcd} \epsilon^{ijk} l^a \partial_i l^b \partial_j l^c \partial_k l^d d^3x. \quad (8)$$

One can see that the integrated function in the right side is the unit surface element of S^3 which implies that Hopf invariant is just the winding number of Gauss mapping $S^3 \rightarrow S^3$. Using ϕ -mapping method the inner topological structure can be studied. While S^3 can be viewed as the infinite boundary of R^4 , using Stokes theorem one can arrive at

$$H = \frac{1}{12\pi^2} \int_{R^4} \epsilon_{abcd} \epsilon^{\mu\nu\lambda\rho} \partial_\mu l^a \partial_\nu l^b \partial_\lambda l^c \partial_\rho l^d d^4x,$$

where x^μ is the coordinate of R^4 . According to ϕ -mapping topological theory[13], one can use Eq.(7) and the Green function relation in ϕ -space

$$\frac{\partial^2}{\partial \varphi^a \partial \varphi^a} \left(\frac{1}{\|\varphi\|} \right) = -4\pi^2 \delta^4(\vec{\varphi})$$

to express Hopf invariant in the following form

$$H = \int_{R^4} \delta^4(\vec{\varphi}) D\left(\frac{\varphi}{x}\right) d^4x, \quad (9)$$

where $D(\frac{\varphi}{x}) = (1/4!) \epsilon_{abcd} \epsilon^{\mu\nu\lambda\rho} \partial_\mu \varphi^a \partial_\nu \varphi^b \partial_\lambda \varphi^c \partial_\rho \varphi^d$ is the Jacobian determinant. Suppose $\varphi^a(x) (a = 0, 1, 2, 3)$ have m isolated zeros at $x^\mu = x_i^\mu (i = 1, 2, \dots, m)$. According to δ -function theory[22], $\delta^4(\vec{\varphi})$ can be expressed by

$$\delta^4(\vec{\varphi}) = \sum_{i=1}^m \frac{\beta_i \delta^4(x - x_i)}{|D(\frac{\varphi}{x})|_{x=x_i}},$$

where β_i is the Hopf index of the i th zero. In topology it means that when the point x^μ covers the neighborhood of the zero point x_i^μ once, the vector field φ^a covers the corresponding region in φ -space β_i times. Substituting the expanding of $\delta^4(\vec{\varphi})$ into Eq.(9), one can get

$$H = \sum_{i=1}^m W_i, \quad (10)$$

where $W_i = \beta_i \eta_i$ the winding number of Gauss mapping $z : S^3 \rightarrow S^3$ with $\eta_i = \text{sign}[D(\phi/x)]_{x_i^\mu} = \pm 1$ is the Brouwer degree.

In this section, we have applied the spinor representation of Hopf map to calculate the Hopf invariant, and reveal its inner topological structure. Although Hopf map $S^3 \rightarrow S^2$ arises in many contexts, Hopf invariant do not depend on the special representation of Hopf map because of the character of topological invariance. As discussed in this section, Hopf invariant is just the winding number of Gauss map $S^3 \rightarrow S^3$ which means that Hopf invariant is equal to the homotopy group $\pi_3(S^3)$. In another aspect, Hopf invariant is characterized by the homotopy group $\pi_3(S^2)$. The above considerations lead us to the following crucial relation $\pi_3(S^3) = \pi_3(S^2) = \mathbb{Z}$. This can be interpreted as following. In fact, the definition of \vec{m} involves a two-step map $S^3 \rightarrow S^3 \rightarrow S^2$. The first step is given by the spinor $z : S^3 \rightarrow S^3$, and the second step is given by the unit vector \vec{m} . The first step of the map is an Gauss map. We have discussed that Hopf invariant is defined as the winding number of the first step map $z : S^3 \rightarrow S^3$. In algebraic topology, this result is obtained using the machinery of the exact homotopy sequences[20] of fibre bundles.

III. THE INNER STRUCTURE OF TOPOLOGICAL CURRENT

We find that, from the Hopf curvature (4), a conserved topological current can be naturally introduced. In this section, we mainly study the inner structure of this conserved

topological current. This section plays a crucial role in establishing the relationship between Hopf invariant and linking number of the knot family.

According to ϕ -mapping topological current theory, we can define a topological current

$$j^i = \frac{1}{8\pi} \epsilon^{ijk} B_{jk} = \frac{1}{8\pi} \epsilon^{ijk} \epsilon_{abc} m^a \partial_j m^b \partial_k m^c. \quad (11)$$

It can be proved that

$$B_{jk} = \epsilon_{abc} m^a \partial_j m^b \partial_k m^c = 2\epsilon_{ab} \partial_j n^a \partial_k n^b,$$

where $n^a (a = 1, 2)$ is a two-dimensional vector in the tangent space of sphere S^2 . The above relation is known as Mermin-Ho relation[23]. The two-dimensional vector n^a is defined as

$$n^a = \frac{\phi^a}{\|\phi\|}, \|\phi\| = \phi^a \phi^a, a = 1, 2. \quad (12)$$

where $\vec{\phi}$ is a two-dimensional vector function on R^3 . Then the topological current can be expressed as

$$j^i = \frac{1}{4\pi} \epsilon^{ijk} \epsilon_{ab} \partial_j n^a \partial_k n^b = \delta^2(\vec{\phi}(x)) D^i\left(\frac{\phi}{x}\right), \quad (13)$$

where $D^i(\frac{\phi}{x}) = \frac{1}{2} \epsilon^{ijk} \epsilon_{ab} \partial_j \phi^a \partial_k \phi^b$ is the Jacobian vector. This expression of j^i provides an important conclusion

$$j^i \begin{cases} = 0 & \text{if and only if } \vec{\phi} \neq 0, \\ \neq 0 & \text{if and only if } \vec{\phi} = 0. \end{cases}$$

So it is necessary to study the zero points of $\vec{\phi}$ to determine the nonzero solution of j^i . The implicit function theory[24] show that, under the regular condition $D^i(\frac{\phi}{x}) \neq 0$, the general solutions of

$$\phi^1(x^1, x^2, x^3) = 0, \phi^2(x^1, x^2, x^3) = 0,$$

can be expressed as

$$x^1 = x_k^1(s), x^2 = x_k^2(s), x^3 = x_k^3(s).$$

which represent N isolated singular strings $L_k (k = 1, 2, \dots, N)$ with string parameter s .

In δ -function theory[22], one can prove that in three dimension space the expanding of $\delta^2(\vec{\phi})$ is

$$\delta^2(\vec{\phi}) = \sum_{k=1}^N \beta_k \int_{L_k} \frac{\delta^3(\vec{x} - \vec{x}_k(s))}{|D(\frac{\phi}{u})|_{\Sigma_k}} ds, \quad (14)$$

where $D(\frac{\phi}{u}) = \frac{1}{2}\epsilon^{ij}\epsilon_{ab}\frac{\partial\phi^a}{\partial u^i}\frac{\partial\phi^b}{\partial u^j}$ and Σ_k is the k th planar element transverse to L_k with local coordinates (u^1, u^2) . The positive integer β_k is the Hopf index of ϕ mapping. Meanwhile the tangent vector of L_k is given by

$$\frac{dx^i}{ds} \big|_{L_k} = \frac{D^i(\frac{\phi}{x})}{D(\frac{\phi}{u})} \big|_{L_k}.$$

Then the inner topological structure of j^i is

$$j^i = \sum_{k=1}^N W_k \int_{L_k} \frac{dx^i}{ds} \delta^3(\vec{x} - \vec{z}_i(s)) ds, \quad (15)$$

where $W_k = \beta_k \eta_k$ is the winding number of $\vec{\phi}$ around L_k , with $\eta_k = \text{sgn} D(\frac{\phi}{u})|_{\Sigma_k} = \pm 1$ being the Brouwer degree of ϕ mapping.

It can be seen that when these singular strings are closed curves or more generally are a family of N knots $\gamma_k (k = 1, 2, \dots, N)$, the inner structure of topological current is

$$j^i = \sum_{k=1}^N W_k \oint_{\gamma_k} \frac{dx^i}{ds} \delta^3(\vec{x} - \vec{z}_i(s)) ds, \quad (16)$$

and Hopf invariant can be written as

$$H = \frac{1}{4\pi} \sum_{k=1}^N W_k \oint_{\gamma_k} A_i dx^i. \quad (17)$$

From the discussion of Sec.II, we see that A_i is similar with the vector potential in gauge theory. But the expression (17) indicates that it is only for closed strings that Hopf invariant is unchanged under the $U(1)$ gauge transformation of A_i . This is the reason why we only consider the closed configuration of these singular strings.

IV. THE PRECISE EXPRESSION FOR HOPF INVARIANT

One can find that it is useful to express A_i in terms of the topological current j^i . We know that A_i is in the form of $U(1)$ gauge potential in $U(1)$ electromagnetic gauge theory.

So the topological current j^i is similar with magnetic field and the method in classical electrodynamics should be useful. Now we can introduce a vector as

$$C_i = \epsilon_{ijk} \partial_j j^k. \quad (18)$$

As pointed out in Sec.II, one can impose the condition $\partial_i A_i = 0$ which is to say that we select the Columb gauge in the discussion. It is easy to get

$$C_i = -\frac{1}{4\pi} \partial_j \partial_j A_i.$$

This is just the Possion equation. The general solution of the above equation is

$$A_i = - \int d^3 y \frac{C_i(y)}{|\vec{x} - \vec{y}|}.$$

Then we obtain the crucial relation between A_i and the topological current j^i

$$A_i = -\epsilon_{ijk} \int d^3 y \frac{\partial_j j^k(y)}{|\vec{x} - \vec{y}|}. \quad (19)$$

Then the Hopf invariant is

$$\begin{aligned} H &= \frac{1}{4\pi} \int A_i j^i d^3 x \\ &= -\frac{1}{4\pi} \epsilon_{ijk} \int d^3 x \int d^3 y j^i(x) \partial_j j^k(y) \frac{1}{|\vec{x} - \vec{y}|}. \end{aligned}$$

Because the integral is on the total space, under the boundary condition (2) the topological current vanishes naturally on the boundary. Then one can get

$$H = \frac{1}{4\pi} \epsilon_{ijk} \int d^3 x \int d^3 y j^i(x) j^j(y) \partial_k \frac{1}{|\vec{x} - \vec{y}|}.$$

Substituting the inner structure of topological current (16) into the above equation, one can get

$$H = \frac{1}{4\pi} \sum_{m,n=1}^N W_m W_n \epsilon_{ijk} \oint_{\gamma_m} dx^i \oint_{\gamma_n} dy^j \partial_k \frac{1}{|\vec{x} - \vec{y}|}. \quad (20)$$

One should notice that the above equation includes two cases:(1) γ_m and γ_n are two different knots($m \neq n$). Noticing that the Gauss linking number is defined as

$$\mathcal{L}k(\gamma_m, \gamma_n) = \frac{1}{4\pi} \epsilon_{ijk} \oint_{\gamma_m} dx^i \oint_{\gamma_n} dy^j \partial_k \frac{1}{|\vec{x} - \vec{y}|}, \quad (21)$$

we can get a portion of Hopf invariant

$$H^{(1)} = \sum_{m,n=1, m \neq n}^N W_m W_n \mathcal{L}k(\gamma_m, \gamma_n).$$

(2) γ_m and γ_n are the same knots. It can be proved that

$$\mathcal{L}k(\gamma_m, \gamma_n) = \mathcal{S}\mathcal{L}(\gamma_m) = \mathcal{T}w(\gamma_m) + \mathcal{W}r(\gamma_m),$$

where $\mathcal{S}\mathcal{L}(\gamma_m)$ is the self-linking number of γ_m . This formula is well known as White formula[26] with $\mathcal{T}w(\gamma_m)$ and $\mathcal{W}r(\gamma_m)$ being the twisting number and writhing number of γ_m respectively. So we get another portion of Hopf invariant

$$H^{(2)} = \sum_{m=1}^N W_m^2 \mathcal{S}\mathcal{L}(\gamma_m).$$

Finally, we add the two parts up and get an important expression for Hopf invariant

$$H = \sum_{m=1}^N W_m^2 \mathcal{S}\mathcal{L}(\gamma_m) + \sum_{m,n=1, m \neq n}^N W_m W_n \mathcal{L}k(\gamma_m, \gamma_n). \quad (22)$$

This precise expression reveals the relationship between the Hopf invariant and the self-linking and linking numbers of N knots family. Since the self-linking and linking numbers are both invariant characteristic numbers of the knotlike closed curves in topology, the Hopf invariant is an important invariant required to describe the knotlike configurations in physics.

V. CONCLUSION

In this paper, in light of ϕ -mapping topological current theory, the inner structure of Hopf invariant is studied in detail. It is revealed that Hopf invariant is just the winding number of Gauss mapping of 3-sphere. we also introduce a conserved topological current from which a family of knots can be deduced naturally. According to the inner structure of topological current, a precise expression for Hopf invariant is presented. It is the total sum of all the self-linking and all the linking numbers of the knot family. A trivial generation of Hopf invariant from $\pi_3(S^2)$ to $\pi_{2n-1}(S^n)$ (where n is positive integer) is valuable for further study.

At last, to complete this paper, a final remark is necessary. An interesting question arise that the magnetic field may be described by the topological current. Because the magnetic force lines are closed in the case that there is no magnetic monopole existed in the spacetime, the topology and geometry of magnetic field can be studied by the ϕ -mapping topological current theory. Recently, Faddeev and Niemi have constructed a model[27] to describe the knotted and linked configuration of electrical neutral plasmas, which may be provide us a theoretical framework to study this question.

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